

Simple groups:

Def: a group G is called **simple** if it has exactly 2 normal subgroups (1 and G)

only quotients are $G/1 \cong G$ and $G/G \cong 1$

Ex: assume G abelian is simple;

take any $e \neq g \in G$; consider $\mathbb{Z}/n\mathbb{Z} \cong H \subseteq G$ generated by g

$H = G$, so G is cyclic

- \mathbb{Z} is not simple (it has a ton of subgroups $\cong \mathbb{Z}$)
- $\mathbb{Z}/n\mathbb{Z}$ is simple $\iff n$ prime

Non-Ex: \mathbb{Z} is not simple

$G \times H$ where $G, H \neq \{1\}$

Thm: $\forall n \geq 5$, the **alternating group**

$$A_n = \text{Ker} \left(S_n \xrightarrow{\text{sign}} \mathbb{Z}/2\mathbb{Z} \right)$$

is simple.

Why do we care: suppose you have a **machine** which allows you to reconstruct a group G from a normal subgroup $H \triangleleft G$ and the quotient G/H

homological algebra

Then simple groups are the building blocks of many groups

$$\bullet G = \mathbb{Z}/6\mathbb{Z} \rightsquigarrow \begin{array}{l} H = 3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \\ H' = 2\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \end{array} \left| \begin{array}{l} G/H \cong \mathbb{Z}/3\mathbb{Z} \\ G/H' \cong \mathbb{Z}/2\mathbb{Z} \end{array} \right.$$

$$0 \rightarrow \overset{H}{\mathbb{Z}/2\mathbb{Z}} \rightarrow G \rightarrow \overset{G/H}{\mathbb{Z}/3\mathbb{Z}} \rightarrow 0$$

$$1 = G_0 \triangleleft \overset{H}{G_1} \triangleleft G_2 = G$$

$$0 \rightarrow \underset{H'}{\mathbb{Z}/3\mathbb{Z}} \rightarrow G \rightarrow \underset{G/H'}{\mathbb{Z}/2\mathbb{Z}} \rightarrow 0$$

$$1 = G_0 \triangleleft \underset{H'}{G_1} \triangleleft G_2 = G$$

Def: given a group G , a **subnormal series**

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$$

k = length of series

s.t. G_{i-1} is a normal subgroup of G_i , $\forall i$

(a **normal series** is when G_{i-1} is normal in G , $\forall i$)

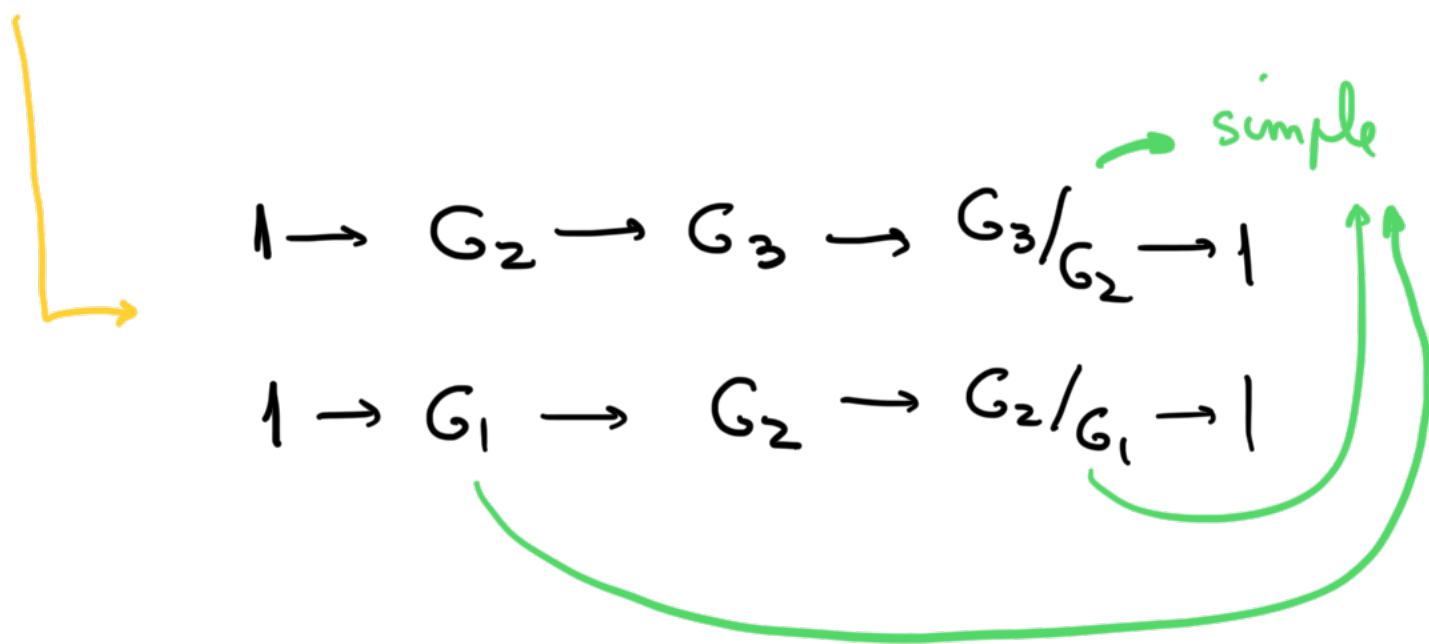
\triangleright A **Sylow series** is a subnormal series s.t. G_i/G_{i-1} is simple

Def: A subnormal series $\dots \triangleleft G_{i-1}$

is called a **composition series**

If G admits a composition series, then we can reconstruct it from simples using the **green machine**,

e.g. $1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft G_3 = G$, then



• \mathbb{Z} does not admit a composition series

• $\mathbb{Z}/6\mathbb{Z}$ does admit a composition series

• S_n admits a composition series

$$\left(1 \triangleleft A_n \triangleleft S_n \right)_{n \geq 5}$$

$A_n/1$ simple

$S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$

Prop: a subnormal series is a composition series if and only if it is **maximal**

cannot extend to a longer subnormal series

Proof: suppose \exists subnormal series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$

which is maximal; if G_i/G_{i-1} were not simple, then

$$\exists 1 \triangleleft H \triangleleft G_i/G_{i-1}$$

\rightsquigarrow
correspondence
theorem

$$G_{i-1} \triangleleft \tilde{H} \triangleleft G_i$$

\Downarrow
contradicts maximality of
the subnormal series

Thm: any finite group has a composition series

Proof: just take a maximal subnormal series, which exists because a finite group has finitely many subgroups (start from $1 \triangleleft G$ and extend it as long as you can by inserting subgroups)

Normal subgroups & quotient groups *inherit* composition series

Prop 1: \forall composition series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$
($H \triangleleft G$) consider $1 = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_{k-1} \triangleleft H_k = H$

with $H_i = H \cap G_i$

after removing redundancies (i.e. if $H_{i-1} = H_i$, remove one of them) this is a composition series of H .

Prop 2: \forall composition series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$

$(H \triangleleft G)$

consider

$$1 = \bar{G}_0 \triangleleft \bar{G}_1 \triangleleft \dots \triangleleft \bar{G}_{k-1} \triangleleft \bar{G}_k = \bar{G}$$

$$\bar{G} = G/H$$

with $\bar{G}_i = HG_i/H$

after removing redundancies (i.e. if $\bar{G}_{i-1} = \bar{G}_i$ remove one of them) this is a composition series of \bar{G} .

Proof of Prop 1: $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$

\triangleleft
 H

let $H_i = H \cap G_i$, must check two things:

- H_{i-1} is normal in H_i : $\forall g \in H_i, h \in H_{i-1}$, have $ghg^{-1} \in H$ b/c $h, g \in H$
 $\in G_{i-1}$ b/c $h \in G_{i-1}$
 $g \in G_i$
 \Downarrow
 $ghg^{-1} \in H_{i-1} \checkmark$

- | | | |
|---------------------------|-----------------------|--|
| $H_i \subset G_i$ | <u>induced hom</u> | $H_i/H_{i-1} \xrightarrow{\phi} G_i/G_{i-1}$ |
| \triangleleft | $\xrightarrow{\quad}$ | $[h] \rightsquigarrow [h]$ |
| $H_{i-1} \subset G_{i-1}$ | | |

ϕ is a injective homomorphism b/c $\phi(h) = e$



$$\begin{aligned} & \Downarrow \\ & h \in G_{i-1} \Rightarrow h \in H_{i-1} \\ & h \in H_i \end{aligned}$$

H_i/H_{i-1} is a subgroup of G_i/G_{i-1}

and it's normal b/c $H \triangleleft G \implies H_i \triangleleft G_i \implies H_i/H_{i-1} \triangleleft G_i/G_{i-1}$
↓
simple

$\implies H_i/H_{i-1} = \{1\} \rightsquigarrow$ redundancy

$H_i/H_{i-1} \cong G_i/G_{i-1}$ and hence it's simple \square

length of $1=H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_{k-1} \triangleleft H_k=H$ after removing redundancies
 is $<$ length of $1=G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k=G$, if $H \triangleleft G$

\uparrow
 if $H \triangleleft G$, \exists at least one redundancy.

if \exists redundancies, then $H_i/H_{i-1} \xrightarrow{\phi} G_i/G_{i-1}$ would be isomorphism, $\forall i$

\downarrow
 by induction on i , we would conclude $H=G$

Proof of Prop 2:

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$$

$$\begin{pmatrix} H \triangleleft G \\ G \twoheadrightarrow \bar{G} = G/H \end{pmatrix}$$

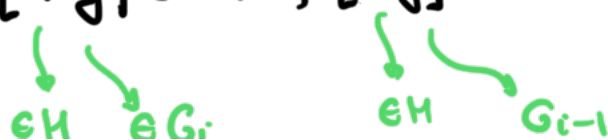
$$\downarrow$$

$$\bar{G}$$

let $\bar{G}_i = HG_i/H$ (H normal $\implies HG_i$ is a subgroup of G)

SII 2nd isomorphism theorem
 $G_i/H \cap G_{i-1}$

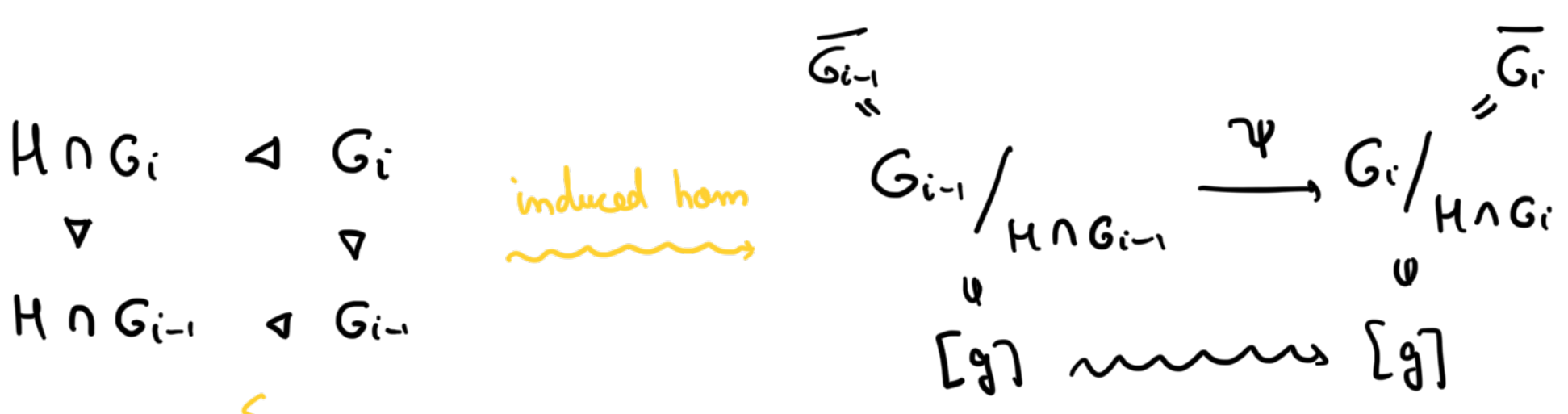
\bar{G}_{i-1} is normal in \bar{G}_i : $[hg] \in \bar{G}_i$, $[h'g'] \in \bar{G}_{i-1}$



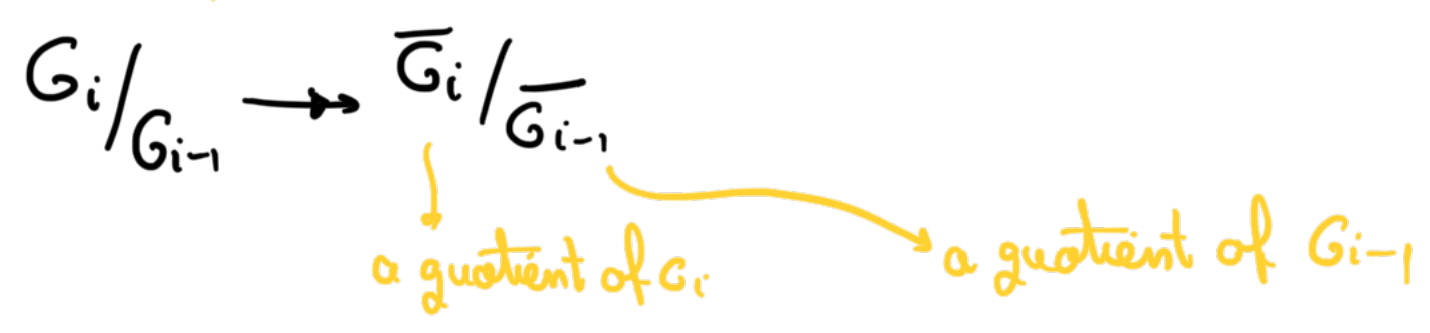
equivalence classes mod h .

$$[hg][h'g'][hg]^{-1} = [hg \underbrace{h'g'}_{h''g} g^{-1} h^{-1}] \in [H g g' g^{-1} H] = [H g g' g^{-1}] \in \overline{G_{i-1}}$$

$h''g, \text{ for some } h'' \in H \quad (\text{b/c } gH = Hg)$



ψ is injective \downarrow b/c $\psi([g]) = e \Rightarrow g \in H \cap G_i \Rightarrow g \in H \cap G_{i-1} \Rightarrow [g] = e$



but $G_i /_{G_{i-1}}$ is simple, so either

$$\begin{cases} \overline{G_i} /_{\overline{G_{i-1}}} = 1 & (\text{redundancy}) \\ \overline{G_i} /_{\overline{G_{i-1}}} \cong G_i /_{G_{i-1}} & \text{is simple} \end{cases}$$

Prop: $1 \longrightarrow K \longrightarrow G \longrightarrow L \longrightarrow 1$

assume \mathcal{F} composition series $1 = K_0 \triangleleft K_1 \triangleleft \dots \triangleleft K_m = K$
 $1 = L_0 \triangleleft L_1 \triangleleft \dots \triangleleft L_n = L$

then \mathcal{F} composition series of length $m+n$ of G

Proof: by the correspondance theorem, we have \rightsquigarrow below

$$\begin{array}{ccc}
 G/K \cong L = L_n & \rightsquigarrow & K < \tilde{L}_n = G \\
 \downarrow \vdots & & \downarrow \vdots \\
 L_i & \rightsquigarrow & K < \tilde{L}_i < G \\
 \downarrow & & \downarrow \\
 L_{i-1} & \rightsquigarrow & K < \tilde{L}_{i-1} < G \\
 \downarrow \vdots & & \downarrow \vdots \\
 1 = L_0 & \rightsquigarrow & K = \tilde{L}_0 < G
 \end{array}$$

So $1 = K_0 \triangleleft K_1 \triangleleft \dots \triangleleft K_m = K = \tilde{L}_0 \triangleleft \tilde{L}_1 \triangleleft \dots \triangleleft \tilde{L}_n = G$ is a composition series of G

Jordan-Hölder theorem: suppose G has two composition series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G = G'_L \triangleleft G'_{L-1} \triangleleft \dots \triangleleft G'_1 \triangleleft G'_0 = 1$$

These two composition series are **equivalent**, i.e. $k=L$

$$\text{and } \left\{ G_1/G_0, \dots, G_k/G_{k-1} \right\} \stackrel{\text{up to iso and permutation}}{=} \left\{ G'_1/G'_0, \dots, G'_L/G'_{L-1} \right\}$$

↓ composition factors

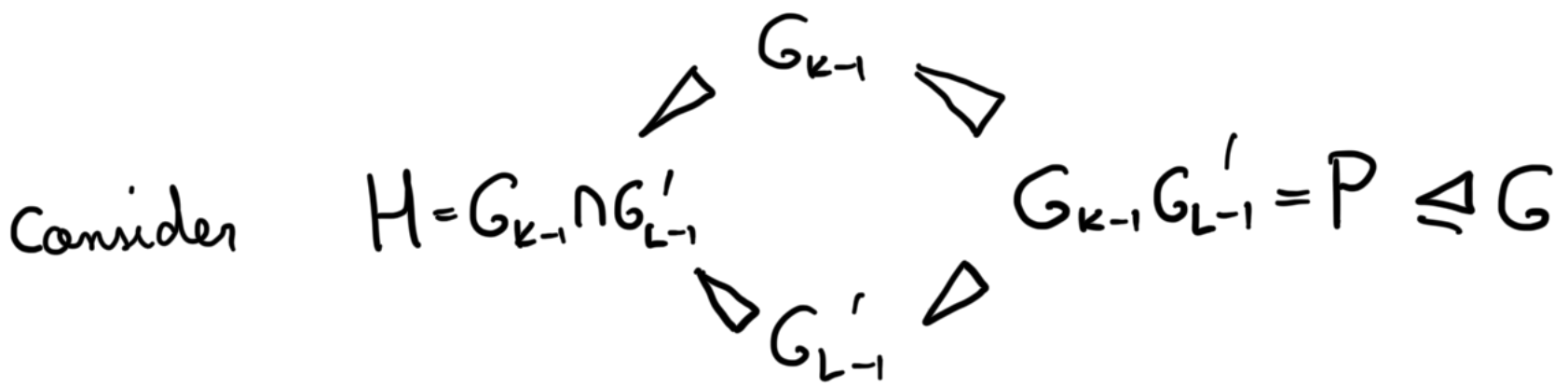
D. A. I. induction on $\min(k, L)$ breaks ties with $k+L$

1 proof: by induction on (k, l)

base case: $k=1$, l is arbitrary

G is simple $\Rightarrow L=1$ and both composition series are $1 \triangleleft G$.

induction step: we may assume $G_{k-1} \neq G'_{l-1}$, or else we could just apply induction hypothesis to G_{k-1} instead of G



2nd iso theorem: $G_{k-1}/H \cong P/G'_{l-1} \triangleleft G/G'_{l-1}$
 $G'_{l-1}/H \cong P/G_{k-1} \triangleleft G/G_{k-1}$ $\Rightarrow P=G$
 (or else $G_{k-1} = P = G'_{l-1}$)

$G_{k-1}/H \cong G/G'_{l-1}$ simple \odot

$G'_{l-1}/H \cong G/G_{k-1}$ simple

$H \triangleleft G_{k-1}$ $\xrightarrow{\text{Prop 1}}$ inherits a composition series from G_{k-1}

$$1 = H'_0 \triangleleft H'_1 \triangleleft \dots \triangleleft H'_m = H, \quad m < k-1$$

$H \triangleleft G_{L-1}' \xrightarrow{\text{Prop 1}}$ inherits a composition series from G_{L-1}'
 $1 = H_0'' \triangleleft H_1'' \triangleleft \dots \triangleleft H_m'' = H, m < L-1$

$$\begin{array}{c}
 1 = H_0' \triangleleft H_1' \triangleleft \dots \triangleleft H_m' = H \triangleleft G_{k-1} \triangleleft G \\
 \underbrace{\hspace{10em}}_{\text{equiv by ind hyp}} \quad \quad \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \\
 1 = H_0'' \triangleleft H_1'' \triangleleft \dots \triangleleft H_m'' = H \triangleleft G_{L-1}' \triangleleft G \\
 \quad \quad \quad \begin{array}{c} \nwarrow \\ \swarrow \end{array}
 \end{array}$$

The green composition factors are \cong by \otimes

\Downarrow
 The above composition series are equivalent

But also, the top composition series is equivalent to

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-2} \triangleleft G_{k-1} \triangleleft G \quad (\text{ind. hyp for } G_{k-1})$$

while the bottom composition series is equivalent to

$$1 = G_0' \triangleleft G_1' \triangleleft \dots \triangleleft G_{L-2}' \triangleleft G_{L-1}' \triangleleft G \quad (\text{ind. hyp for } G_{L-1}')$$

By transitivity of equivalence, these two are also equivalent