Simple groups:

<u>Def</u>: a group G is called simple if it has exactly 2 normal subgroups (1 and G)

$\Updownarrow$

only quotients are $G/_1 \cong G$ and $G/_G \cong 1$

Ex: assume G abelian is simple;

take any $e \neq g \in G$; consider $\mathbb{Z}/_{n\mathbb{Z}} \cong H \subseteq G$ generated by g

$\Downarrow$

$H = G$, so G is cyclic

- $\mathbb{Z}$ is not simple (it has a ton of subgroups $\cong \mathbb{Z}$)
- $\mathbb{Z}/_{n\mathbb{Z}}$ is simple $\Longleftrightarrow$ n prime

Non-Ex: $\mathbb{Z}$ is not simple

$G \times H$ where $G, H \neq \{1\}$

<u>Thm</u>: $\forall\ n \geq 5$, the alternating group

$$A_n = \mathrm{Ker}\left(S_n \xrightarrow{\text{sign}} \mathbb{Z}/_{2\mathbb{Z}}\right)$$

is simple

is simple.

Why do we care: suppose you have a machine which allows you to reconstruct a group G from a normal subgroup $H \trianglelefteq G$ and the quotient G/H

machine $\leftarrow$ homological algebra

Then simple groups are the building blocks of many groups

- $G = \mathbb{Z}/6\mathbb{Z} \rightsquigarrow$

$$H = 3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \qquad G/H \cong \mathbb{Z}/3\mathbb{Z}$$

$$H' = 2\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \qquad G/H' = \mathbb{Z}/2\mathbb{Z}$$

$$0 \to \underset{H}{\mathbb{Z}/2\mathbb{Z}} \to G \to \underset{G/H}{\mathbb{Z}/3\mathbb{Z}} \to 0 \qquad 1 = G_0 \trianglelefteq \underset{H}{G_1} \trianglelefteq G_2 = G$$

$$\leftrightsquigarrow$$

$$0 \to \underset{H'}{\mathbb{Z}/3\mathbb{Z}} \to G \to \underset{G/H'}{\mathbb{Z}/2\mathbb{Z}} \to 0 \qquad 1 = G_0 \trianglelefteq \underset{H'}{G_1} \trianglelefteq G_2 = G$$

Def: given a group G, a subnormal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G_{k-1} \trianglelefteq G_k = G$$

k = length of series

s.t. G_{i-1} is a normal subgroup of G_i, $\forall i$

(a normal series is when G_{i-1} is normal in G, $\forall i$)

Def: a subnormal series s.t. G_i/G_{i-1} is simple

Def: a subnormal series s.t. ${}^{G_i}/_{G_{i-1}}$ ... simple
is called a composition series

If G admits a composition series, then we can reconstruct it from simples using the green machine,

e.g. $1 = G_0 \lhd G_1 \lhd G_2 \lhd G_3 = G$, then

$$1 \to G_2 \to G_3 \to {}^{G_3}/_{G_2} \to 1 \quad \text{simple}$$
$$1 \to G_1 \to G_2 \to {}^{G_2}/_{G_1} \to 1$$

- $\mathbb{Z}$ does not admit a composition series
- $\mathbb{Z}/6\mathbb{Z}$ does admit a composition series
- S_n admits a composition series

$$\big(1 \lhd A_n \lhd S_n \big)_{n \geq 5}$$

$A_n/1$ simple $\qquad S_n/A_n \cong \mathbb{Z}/2\mathbb{Z}$

Prop: a subnormal series is a composition series if and only if it is maximal cannot extend to a longer subnormal series

Proof: suppose $\exists$ subnormal series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$ which is maximal; if G_i/G_{i-1} were not simple, then

$\exists \quad 1 \triangleleft H \triangleleft G_i/G_{i-1}$ $\overset{\text{correspondence theorem}}{\rightsquigarrow}$ $G_{i-1} \triangleleft \tilde{H} \triangleleft G_i$

contradicts maximality of the subnormal series

Thm: any finite group has a composition series

Proof: just take a maximal subnormal series, which exists because a finite group has finitely many subgroups (start from $1 \triangleleft G$ and extend it as long as you can by inserting subgroups)

Normal subgroups & quotient groups inherit composition series

Prop 1: $\forall$ composition series $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{k-1} \triangleleft G_k = G$

$(H \triangleleft G)$ consider $1 = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{k-1} \trianglelefteq H_k = H$

with $H_i = H \cap G_i$

after removing redundancies (i.e. if $H_{i-1} = H_i$, remove one of them) this is a composition series of H.

<u>Prop</u> 2: $\forall$ composition series $1 = G_0 \lhd G_1 \lhd \dots \lhd G_{k-1} \lhd G_k = G$

$(H \lhd G)$

$\overline{G} = G/H$

consider $1 = \overline{G}_0 \trianglelefteq \overline{G}_1 \trianglelefteq \dots \trianglelefteq \overline{G}_{k-1} \trianglelefteq \overline{G}_k = \overline{G}$

with $\overline{G}_i = HG_i/H$

after removing redundancies (i.e. if $\overline{G}_{i-1} = \overline{G}_i$ remove one of them) this is a composition series of $\overline{G}$.

Proof of Prop 1: $1 = G_0 \lhd G_1 \lhd \dots \lhd G_{k-1} \lhd G_k = G \rhd H$

let $H_i = H \cap G_i$, must check two things:

- H_{i-1} is normal in H_i: $\forall\, g \in H_i,\ h \in H_{i-1}$, have ghg^{-1} $\in H$ b/c $h, g \in H$; $\in G_{i-1}$ b/c $h \in G_{i-1}$, $g \in G_i$

$$\Downarrow$$

$$ghg^{-1} \in H_{i-1} \checkmark$$

- $$\begin{matrix} H_i & \subset & G_i \\ \triangledown & & \triangledown \\ H_{i-1} & \subset & G_{i-1} \end{matrix}$$

induced hom

$$H_i/H_{i-1} \xrightarrow{\phi} G_i/G_{i-1}$$
$$[h] \rightsquigarrow [h]$$

ϕ is a injective homomorphism b/c $\phi(h) = e$

$$\Updownarrow$$

$h \in G_{i-1}$, $h \in H_i$ $\Rightarrow h \in H_{i-1}$

$$\Downarrow$$

$H_i/_{H_{i-1}}$ is a subgroup of $G_i/_{G_{i-1}}$

and it's normal b/c $H \trianglelefteq G \Rightarrow H_i \trianglelefteq G_i \Rightarrow H_i/_{H_{i-1}} \trianglelefteq G_i/_{G_{i-1}}$

simple

$\Rightarrow H_i/_{H_{i-1}} = \{1\}$ $\rightsquigarrow$ redundancy

or

$H_i/_{H_{i-1}} \cong G_i/_{G_{i-1}}$ and hence it's simple $\square$

(length of $1 = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{k-1} \trianglelefteq H_k = H$ after removing redundancies

is $<$ length of $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_{k-1} \trianglelefteq G_k = G$, if $H \ntrianglelefteq G$

$\Uparrow$

if $H \trianglelefteq G$, $\exists$ at least one redundancy.

if $\nexists$ redundancies, then $H_i/_{H_{i-1}} \xrightarrow{\phi} G_i/_{G_{i-1}}$ would be isomorphism, $\forall i$

$\Downarrow$

by induction on i, we would conclude $H = G$)

Proof of Prop 2: $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_{k-1} \trianglelefteq G_k = G$

$\downarrow$

$\overline{G}$

$\left(\begin{array}{c} H \trianglelefteq G \\ G \twoheadrightarrow \overline{G} = G/_H \end{array}\right)$

let $\overline{G_i} = HG_i/_H$ (H normal $\Rightarrow HG_i$ is a subgroup of G)

$\cong$ 2nd isomorphism theorem

$G_i/_{H \cap G_{i-1}}$

- $\overline{G_{i-1}}$ is normal in $\overline{G_i}$: $[hg] \in \overline{G_i}$, $[h'g'] \in \overline{G_{i-1}}$ equivalence classes mod h.

$\in H$ $\in G_i$ $\in H$ G_{i-1}

$$[hg][h'g'][hg]^{-1} = [h\underbrace{gh'}_{h''g}g'g^{-1}h^{-1}] \in [\overset{\in H}{H}\, gg'g^{-1}H] = [H\, \overset{\in G_{i-1}}{gg'g^{-1}}] \in \overline{G_{i-1}}$$

$h''g$, for some $h'' \in H$ (b/c $gH = Hg$)

$$\begin{array}{ccc} H\cap G_i & \lhd & G_i \\ \triangledown & & \triangledown \\ H\cap G_{i-1} & \lhd & G_{i-1} \end{array}$$

induced hom

$$\begin{array}{ccc} \overline{G_{i-1}} = G_{i-1}/H\cap G_{i-1} & \xrightarrow{\psi} & G_i/H\cap G_i = \overline{G_i} \\ [g] & \rightsquigarrow & [g] \end{array}$$

ψ is injective b/c $\psi([g]) = e \Rightarrow \begin{matrix} g \in H\cap G_i \\ g \in G_{i-1} \end{matrix} \Rightarrow g \in H \cap G_{i-1} \Rightarrow [g] = e$

$$G_i/G_{i-1} \twoheadrightarrow \overline{G_i}/\overline{G_{i-1}}$$

a quotient of G_i — a quotient of G_{i-1}

but G_i/G_{i-1} is simple, so either $\begin{cases} \overline{G_i}/\overline{G_{i-1}} = 1 & \text{(redundancy)} \\ \overline{G_i}/\overline{G_{i-1}} \cong G_i/G_{i-1} & \text{is simple} \end{cases}$

<u>Prop</u>: $1 \longrightarrow K \longrightarrow G \longrightarrow L \longrightarrow 1$

assume $\exists$ composition series $1 = K_0 \lhd K_1 \lhd \dots \lhd K_m = K$

$1 = L_0 \lhd L_1 \lhd \dots \lhd L_n = L$

then $\exists$ composition series of length $m+n$ of G

Proof: by the correspondance theorem, we have $\rightsquigarrow$ below

$$\begin{array}{ccc}
G/K \cong L = L_n & \rightsquigarrow & K < \tilde{L}_n = G \\
\triangledown & & \triangledown \\
\vdots & & \vdots \\
\triangledown & & \triangledown \\
L_i & \rightsquigarrow & K < \tilde{L}_i < G \\
\triangledown & & \triangledown \\
L_{i-1} & \rightsquigarrow & K < \tilde{L}_{i-1} < G \\
\triangledown & & \triangledown \\
\vdots & & \vdots \\
\triangledown & & \triangledown \\
1 = L_0 & \rightsquigarrow & K = \tilde{L}_0 < G
\end{array}$$

So $1 = K_0 \triangleleft K_1 \triangleleft \dots \triangleleft K_m = K = \tilde{L}_0 \triangleleft \tilde{L}_1 \triangleleft \dots \triangleleft \tilde{L}_n = G$ is a composition series of G

Jordan-Hölder theorem: suppose G has two composition series

$$1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_{K-1} \triangleleft G_K = G = G'_L \triangleright G'_{L-1} \triangleright \dots \triangleright G'_1 \triangleright G'_0 = 1$$

These two composition series are equivalent, i.e. $K = L$

and $\underbrace{\{G_1/G_0, \dots, G_K/G_{K-1}\}}_{\text{composition factors}} \underset{\text{up to iso and permutation}}{=\!=} \{G'_1/G'_0, \dots, G'_L/G'_{L-1}\}$

Proof by induction on $\min(K, L)$, break ties with $K+L$

Proof : By induction on $\min(K, L)$,

base case : $K=1$, L is arbitrary

$\Downarrow$

G is simple $\Rightarrow$ $L=1$ and both composition series are $1 \triangleleft G$.

induction step : we may assume $G_{K-1} \neq G'_{L-1}$, or else we could just apply induction hypothesis to G_{K-1} instead of G

Consider

$$\begin{tikzcd}[column sep=small] & G_{K-1} \arrow[dr, phantom, "\triangleleft"] & \\ H = G_{K-1} \cap G'_{L-1} \arrow[ur, phantom, "\triangleleft"] \arrow[dr, phantom, "\triangleleft"] & & G_{K-1} G'_{L-1} = P \trianglelefteq G \\ & G'_{L-1} \arrow[ur, phantom, "\triangleleft"] & \end{tikzcd}$$

2nd iso theorem :

$$G_{K-1}/H \cong P/G'_{L-1} \trianglelefteq G/G'_{L-1}$$
$$G'_{L-1}/H \cong P/G_{K-1} \trianglelefteq G/G_{K-1}$$

$\Rightarrow P = G$ (or else $G_{K-1} = P = G'_{L-1}$)

$\Downarrow$

$G_{K-1}/H \cong G/G'_{L-1}$ simple $\ast$

$G'_{L-1}/H \cong G/G_{K-1}$ simple

$H \triangleleft G_{K-1}$ $\overset{\text{Prop 1}}{\rightsquigarrow}$ inherits a composition series from G_{K-1}

$1 = H'_0 \triangleleft H'_1 \triangleleft \cdots \triangleleft H'_m = H$, $m < K-1$

$H \triangleleft G'_{L-1} \overset{\text{Prop 1}}{\rightsquigarrow}$ inherits a composition series from G'_{L-1}

$$1 = H''_0 \triangleleft H''_1 \triangleleft \cdots \triangleleft H''_m = H \quad , \; m < L-1$$

$$1 = H'_0 \triangleleft H'_1 \triangleleft \cdots \triangleleft H'_m = H \triangleleft G_{K-1} \triangleleft G$$

equiv by ind hyp

$$1 = H''_0 \triangleleft H''_1 \triangleleft \cdots \triangleleft H''_m = H \triangleleft G'_{L-1} \triangleleft G$$

The green composition factors are $\cong$ by $(*)$

$$\Downarrow$$

The above composition series are equivalent

But also, the top composition series is equivalent to

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{K-2} \triangleleft G_{K-1} \triangleleft G \quad (\text{ind. hyp for } G_{K-1})$$

while the bottom composition series is equivalent to

$$1 = G'_0 \triangleleft G'_1 \triangleleft \cdots \triangleleft G'_{L-2} \triangleleft G'_{L-1} \triangleleft G \quad (\text{ind. hyp for } G'_{L-1})$$

By transitivity of equivalence, these two are also equivalent